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ON UNIVERSALITY AND SATURATION IN OCEAN INTERNAL WAVES

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The apparent regularity in energy distribution among internal-wave space and time scales may be a saturation phenomenon. This paper develops a model of the saturated internal-wave field based on the hypothesis that the saturated quantity is Ω^2/N^2, the locally normalized squared vorticity. This quantity, which is a generalized inverse Richardson number for spatially varying flow, is shown in the linearized approximation to be the density of a conserved "excitation" I; the excitation is assumed to be uniformly partitioned among the normal modes in a saturated field.

The resulting energy distribution differs from that of Garrett and Munk in that energy is assigned preferentially to the lowest modes. Among the properties derived for this distribution, and subject to verification by ocean measurements, is a universal law, independent of the stratification profile N(z), for coherence spectra of isopycnals in the spatial domain.

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TABLE OF CONTENTS

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LIST OF FIGURES

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1. INTRODUCTION

Evidence is accumulating that the distribution of ocean internal wave energy among space-time scales may follow a universal law. Garrett and Munk (1972, 1975) have proposed tentative forms for this distribution that agree fairly well with a variety of spatial and temporal measurements taken in several locations. It is tempting to ascribe such a universal distribution, if it indeed exists, to a saturation phenomenon, in which energy in excess of the saturated value at a given wavelength and frequency is quickly dissipated through nonlinear exchange or turbulent breaking.

In another paper, Garrett and Munk (1972 A) explore this idea further. They argue that a saturated distribution will be marginally stable, experiencing instabilities only occasionally to balance the slow input of energy. Borrowing the inverse Richardson number, Ri^{-1}, from the theory of parallel stratified flow as a measure of instability, they find that the values implied by their distribution are of an order that would indeed suggest marginal stability.

Suppose we take this idea a step further: If the limiting distribution is determined by physical saturation such that Ri^{-1}, or some other suitable measure of excitation, attains a uniform, universal statistical level, we ought to be able to use this information to deduce *a priori* the form of the distribution. In this paper, such a saturation hypothesis is proposed and its consequences worked out in detail.

For reasons that will be explained, a modified measure of excitation better suited to internal waves, the *vortex intensity*, is introduced in place of the traditional Ri^{-1}. The vortex intensity resembles Ri^{-1} except that total vorticity is substituted for velocity shear. As will be seen, the assumption of a statistically uniform level of excitation throughout the thermocline leads to an energy distribution that is physically reasonable and not in obvious contradiction to existing spatial and temporal measurements, yet quite different in certain respects from the distribution of Garrett and Munk. The major difference is in the

partitioning of energy among the normal modes or vertical wavenumbers, and this difference is not easily distinguishable observationally except through cross-spectral analysis of multi-depth data. Curiously, the model of uniform vortex intensity developed here yields exact analytic predictions of spatial autospectra and vertical coherence, independent of the density-stratification profile, and on this account the model is very directly testable.

The treatment given here assumes no steady shear current and no rotation (no inertial waves). One horizontal dimension, x, is used for notational convenience, but the generalization to two dimensions is straightforward.

2. Vortex Intensity, Its Properties and Proposed Distribution

LINEARIZED NORMAL MODE EQUATIONS

The field $\zeta(x,z,t)$ of vertical fluid displacement is decomposed in the usual way into normal-mode oscillations of horizontal wavenumber k, phase speed c, and frequency $\omega = ck$,

$$\zeta(x,z,t) = \sum_{k,m} a_m(k)\, \phi_m(z) \exp[ik(x \mp c_m t)], \tag{1}$$

the eigenfunctions ϕ_m satisfying

$$\frac{d^2\phi_m}{dz^2} + (c_m^{-2} N^2(z) - k^2)\, \phi_m = 0 \tag{2}$$

and vanishing at the top and bottom of the water column (Phillips 1966, Ch. 5). It is convenient to regard k as an arbitrary but fixed separation parameter on which the discrete eigenvalues c_m^{-2} and eigenfunctions ϕ_m depend; the eigenfunctions then make up an orthogonal and complete set. We use the normalizing convention

$$\int \phi_m N^2 \phi_n dz = \delta_{mn}, \tag{3}$$

according to which the completeness relation is symbolically

$$N^2(z) \sum_{m=1}^{\infty} \phi_m(z)\phi_m(z') = \delta(z-z'). \tag{4}$$

The associated fields of horizontal velocity u and vertical velocity v are, for each mode,

$$u_m = \pm a_m c_m \phi'_m$$

$$v_m = \mp i\, a_m k\, c_m \phi_m, \tag{5}$$

where ϕ' is shorthand for $d\phi/dz$ and where the factor $\exp[ik(x \mp c_m t)]$ has been absorbed into the mode amplitude a_m for brevity.

We assume that the internal wave fields found in nature are homogeneous and random so that their statistical properties are adequately described by the mean-square mode amplitudes, $<|a_m|^2>$, or more precisely by spectral densities $\underset{\sim}{A}_m$ defined over all wavenumbers,

$$<|a_m|^2> \longrightarrow \underset{\sim}{A}_m(k)dk.$$

Our aim is to determine these amplitude spectra, on the one hand from a principle of saturation, and on the other hand from empirical measurements of associated displacement, current, and energy spectra to which the amplitude spectra are uniquely related.

For example, the total wave energy in a water column of unit surface area, at a given k,

$$E = \frac{1}{2}\int \rho[u^2 + v^2 + N^2\zeta^2]dz, \tag{6}$$

becomes with the aid of Equations (1) through (5)

$$E = \rho \sum_m |a_m|^2 \tag{7}$$

so that the spectral density of energy in each mode is

$$\underset{\sim}{E}_m(k) = \rho\, \underset{\sim}{A}_m(k). \tag{8}$$

VORTEX AMPLITUDE AND INTENSITY

To treat the topic of internal wave excitation as generally as possible, we need an absolute dimensionless measure of local amplitude that plays a role for internal waves analogous to that of slope for surface gravity waves. One candidate is the ratio of current shear to local Väisälä-Brunt frequency, $u'/N(z)$, whose square, the inverse Richardson number Ri^{-1}, is the measure of intensity used by Garrett and Munk to assess the degree of saturation of their proposed energy distribution (1972 A) and earlier by Phillips (1966, Ch. 5) to estimate an upper limit to the possible excitation of the first (interface) mode. This normalized amplitude has a serious defect when applied to waves in a vertical continuum, as can be seen in its normal mode expansion via Eqs. (2) and (5), shown here at a single wavenumber k for simplicity:

$$u'(x,z,t)/N(z) = N^{-1}(z) \sum_m a_m c_m \phi''_m$$

$$= N^{-1}(z) \sum_m a_m c_m [k^2 - c_m^{-2} N^2(z)] \phi_m . \qquad (9)$$

This amplitude becomes infinite in regions of vanishing stratification (except at $k = 0$), regions where the local flow is stable potential flow. The infinities arise because of the k^2 terms in the above expression, a clue that the shear amplitude, borrowed from stability theory for parallel shear flow, has not been properly adapted to flows having horizontal periodicity.

This difficulty vanishes when the shear u' is replaced by total wave vorticity

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}$$

to define a vortex amplitude

$$e \equiv \Omega/N; \tag{10}$$

the expansion of this quantity is

$$e(x,z,t) = N^{-1}(z) \sum_m a_m c_m (k^2 \phi_m - \phi_m'')$$

$$= N(z) \sum_m a_m c_m^{-1} \phi_m, \tag{11}$$

showing that the amplitude is well defined regardless of the detailed shape of the stratification profile. The generalized definition of local intensity is then

$$e^2(x,z,t) = \Omega^2/N^2, \tag{12}$$

which reduces to Ri^{-1} in the long-wave limit, $k \to 0$.

Equation (11) above indicates that the vortex amplitude and intensity are confined to stratified layers even though part of the kinetic energy may reside elsewhere in unstratified regions. Intuitively, the question of instability comes down to the mutual interaction of vortex layers, an interaction in which neighboring unstratified flow can be viewed as playing a more or less passive role. On this basis, the vortex intensity would seem to be a plausible measure of the local tendency to instability.

As a general descriptor of internal wave excitation, the vortex intensity has another very desirable property:

Vortex intensity is a conserved density.

Consider a system of internal waves, again at fixed k for simplicity, with an arbitrary, time-dependent vertical structure. We can define a total excitation I per unit surface area by integrating e^2 over the vertical column,

$$I \equiv \int e^2 dz, \tag{13}$$

a definition analogous to that for total energy, with e replacing displacement and velocity. Squaring Eq. (11), we get

$$I = \int \sum_{m,n} a_m^* a_n c_m^{-1} c_n^{-1} \phi_m N^2 \phi_n dz, \tag{14}$$

and by the orthogonality rule of Eq. (3) we find

$$I = \sum_m |a_m|^2 c_m^{-2} = \text{const}: \tag{15}$$

In the limit of linearized dynamics, the excitation I belonging to a group of waves remains unchanged as the group propagates vertically through different, possibly dissimilar, parts of the thermocline.

The excitation has dimensions of length denoting both the level of intensity and the total depth of the water column involved. Equation (15) suggests that the excitation is partitioned among the modes according to

$$I_m = |a_m|^2 c_m^{-2}, \tag{16}$$

which together with Eq. (7) implies that the mode energy and excitation are related by

$$E_m = \rho c_m^2 I_m. \tag{17}$$

PROPOSED ENERGY PARTITIONING

We now pass to a statistical description and replace a given system of waves by an ensemble of systems in which the mode amplitudes are random in phase and amplitude. The ensemble will be stationary in time (at least with respect to mean squares and products such as energy, excitation, and space-time covariances) if the mode amplitudes are uncorrelated:

$$\langle a_m^* a_n \rangle = \langle |a_m|^2 \rangle \delta_{mn}. \tag{18}$$

Defined thus in terms of mean square amplitudes, the ensemble is formally in equilibrium under linearized dynamics. This should be distinguished from physical equilibrium, which is attained when the mean square amplitudes slowly adjust themselves to bring the generation mechanisms, nonlinear energy exchange among modes and wavenumbers, and dissipation into balance. We assume that the time scale for the processes of physical equilibrium is much longer than the oscillation periods ω_m^{-1} so that the stationary ensemble with linearized dynamics is a useful model of equilibrium over many periods.

Can we bypass the intervening physics and directly characterize the saturated, "fully aroused" internal wave field by means of the vortex amplitude? We have argued that the vortex intensity is a plausible measure of local excitation relative to instability, and have shown that it is the density of an invariant, the so-called excitation I. A reasonable guess is that in a saturated field, the excitation is uniformly distributed in depth and among length scales. We make the following hypothesis:

At a given wavenumber, the excitation is partitioned uniformly among the modes.

This proposed distribution implies the following relation among modal mean-square amplitudes and energies:

$$\langle I_m \rangle = I_o, \; m = 1,2, \ldots \tag{19a}$$

$$\langle |a_m|^2 \rangle = c_m^2 \, I_o, \tag{19b}$$

$$\langle E_m \rangle = \rho c_m^2 \, I_o, \tag{19c}$$

where I_o is the universal excitation per mode.

Total energy is finite.

The total mean energy is

$$\langle E \rangle = \rho I_o \sum_{m=1}^{\infty} c_m^2; \tag{20}$$

now c_m is a declining function of m (see Figure 1), which in the limit of high mode numbers is not badly approximated by the WKB values

$$c_m \simeq \pi^{-1} (m - \tfrac{1}{2})^{-1} \int N(z) dz. \tag{21}$$

Thus $c_m^2 \sim m^{-2}$, and even for infinitely many excited modes, the total energy is finite.

Practically speaking, some sort of mode cutoff is required to keep the total excitation from diverging. Viscosity, if nothing else, will

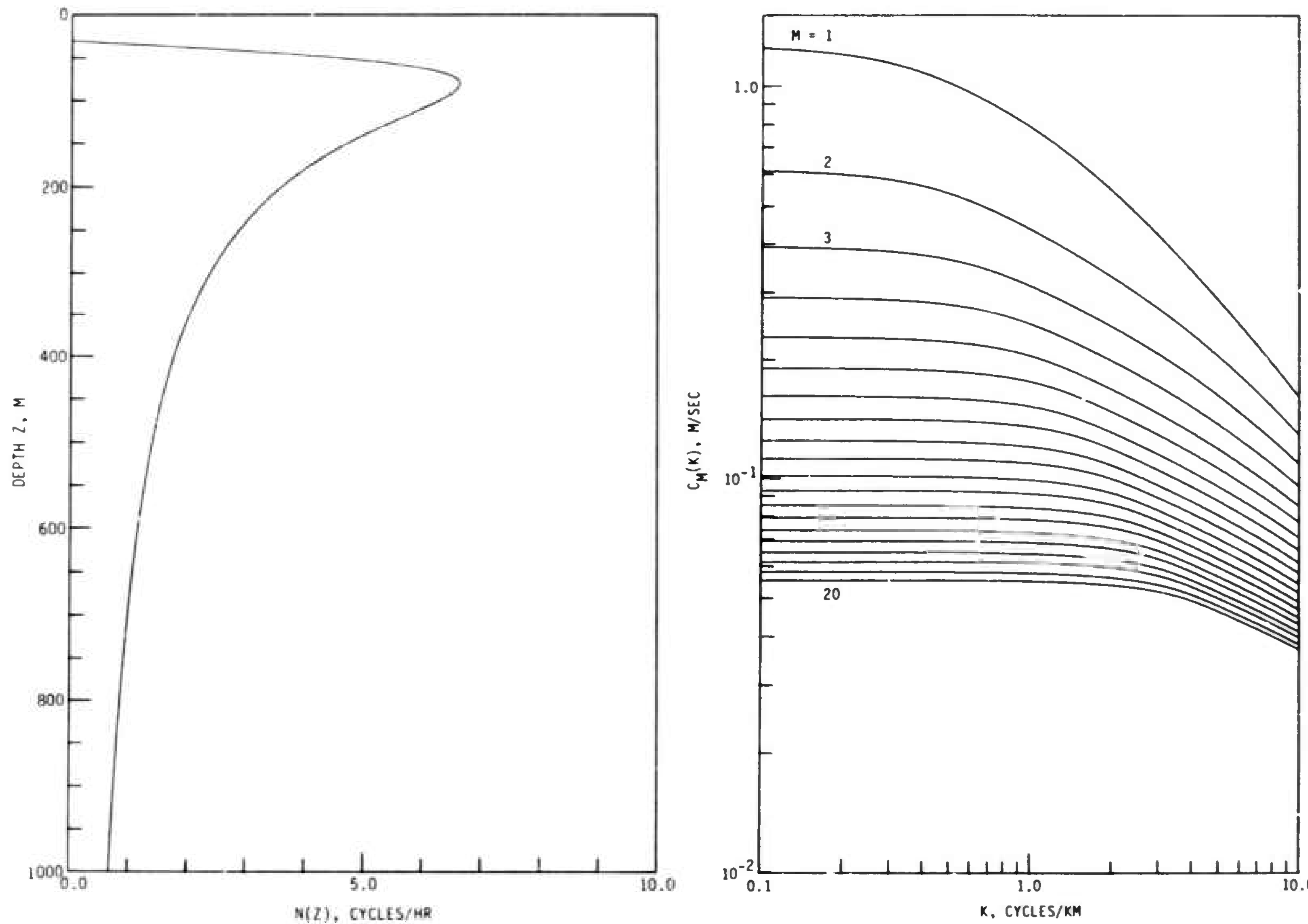

Figure 1. (A) Sample Profile of Väisälä-Brunt Frequency, N(z). (B) Modal Phase Speeds $c_m(k)$ for the Sample Profile. Note that the values become asymptotically constant at low frequency, $kc \ll N_{max}$.

damp modes of vertical scale h where $\omega_m h^2$ is less than the kinematic viscosity. It is important to observe nevertheless that the total energy, and other properties such as predicted covariances of displacement and velocity, will be very weakly dependent on mode cutoff, so that for most purposes the cutoff can be ignored.

The vortex amplitude is statistically uniform in depth and vertical wavenumber.

The two-depth covariance of the random vortex amplitude field e(x,z,t) has a simple form, as deduced from expression (11) via the completeness relation:

$$\langle e(x,z,t)e(x,z',t)\rangle = N^2 \sum_{m,n} \langle a_m^* a_n \rangle c_m^{-1} c_n^{-1} \phi_m(z)\phi_n(z')$$

$$= I_o N^2 \sum_m \phi_m(z)\phi_m(z')$$

$$= I_o \, \delta(z-z'). \qquad (22)$$

Amplitudes at different depths are completely uncorrelated, and the field e is seen to be spectrally white with respect to vertical wavenumber: The vortex amplitudes are in a state of "maximum chaos".

DISTRIBUTION OVER HORIZONTAL SCALES

To generalize the foregoing from a particular horizontal wavenumber to all wavenumbers simultaneously, one merely replaces every mean square quantity by its corresponding spectral density. The fundamental spectrum is then, by hypothesis, the universal excitation spectrum $\underset{\sim}{I}(k)$ and the related mode-amplitude and energy spectra are

$$\underset{\sim}{A}_m(k) = c_m^2(k)\underset{\sim}{I}(k), \tag{23}$$

and

$$\underset{\sim}{E}_m(k) = \rho c_m^2(k)\underset{\sim}{I}(k). \tag{24}$$

Extending the hypothesis that the excitation is partitioned uniformly over all degrees of freedom, we might speculate that $\underset{\sim}{I}(k)$ is white or what we might call "geophysically white", i.e., containing a fixed amount of excitation per octave, so that $\underset{\sim}{I}(k) \sim k^{-1}$. As Phillips found earlier for a hypothesized universal Richardson spectrum, this power law is not steep enough to account for existing observations. A power law closer to $k^{-1.5}$ or k^{-2} appears to be necessary, as will be shown in the next section.

Therefore, at the outset, we have to agree to leave $\underset{\sim}{I}(k)$ somewhat arbitrary, applying the uniform excitation principle only to the vertical degrees of freedom. The dimensionality of the candidate universal spectrum is then reduced to one, and the spectrum can be empirically matched to spatial (tow) data at single depths.

3. Some Consequences of the Proposed Distribution

A uniformly excited internal-wave field will have certain observable characteristics that make it easy to distinguish by means of the appropriate measurements. As one might guess, the most definitive measurements will be those capable of directly resolving horizontal wavenumber, i.e., multi-depth towed measurements, since the proposed distribution assigns energy over spatial scales. Temporal data from moored or floating multi-depth measurements can provide additional checks, but as will be seen, probably cannot by themselves confirm the distributions.

A striking and unexpected property of the uniformly excited field is the existence of simple closed-form predictions, independent of $N(z)$, for spatial autospectra and vertical coherence spectra for the quantity ζ, vertical displacement.

SPATIAL AUTO- AND CROSS-SPECTRA

The instantaneous values of a random displacement field ζ can be expressed formally as an integral over wavenumber of the stochastic amplitudes $a_m(k)$,

$$\zeta(x,z) = \sum_m \int a_m(k)\phi_m(z,k)\exp[ikx]dk, \qquad (25)$$

where the amplitudes have spectral densities $\underset{\sim}{A}_m(k)$ and are mutually uncorrelated at different k and m:

$$\langle a_m^*(k)a_n(k')\rangle = \underset{\sim}{A}_m(k)\delta_{mn}\delta(k-k') \qquad (26)$$

It follows that the cross spectrum of ζ at two depths, z_1 and z_2, is given by

$$\underset{\sim}{Z}_{12}(k) = \sum_m \underset{\sim}{A}_m(k)\phi_m(z_1)\phi_m(z_2) \tag{27}$$

with the single-depth autospectrum represented by the special case $z_2 = z_1$:

$$\underset{\sim}{Z}_1(k) = \sum_m \underset{\sim}{A}_m(k)\phi_m^2(z_1). \tag{28}$$

By Eq. (23) for uniform excitation, $\underset{\sim}{A}_m(k) = c_m^2(k)\underset{\sim}{I}(k)$, we then have

$$\underset{\sim}{Z}_{12}(k) = \underset{\sim}{I}(k) \sum_m c_m^2\phi_m(z_1)\phi_m(z_2). \tag{29}$$

This is an equation for the (symmetric) dependence of $\underset{\sim}{Z}_{12}$ on the two arguments z_1 and z_2 at each k. If we hold z_2 fixed and differentiate twice with respect to z_1, we get via Eq. (2)

$$\frac{\partial^2}{\partial z_1{}^2} \underset{\sim}{Z}_{12} = \underset{\sim}{I} \sum_m c_m^2\phi_m(z_2)(k^2-c_m^{-2})\phi_m(z_1),$$

and subtracting k^2 times $\underset{\sim}{Z}_{12}$,

$$\left(\frac{\partial^2}{\partial z_1{}^2} - k^2\right) \underset{\sim}{Z}_{12} = - \underset{\sim}{I}\, N^2(z_1) \sum_m \phi_m(z_2)\phi_m(z_1)$$

$$= - \underset{\sim}{I}\, \delta(z_1-z_2). \tag{30}$$

Now since ζ vanishes at the top and bottom of the water column, the same must hold for $\underset{\sim}{Z}_{12}$, so that Eq. (30) has the unique solution

$$\underset{\sim}{Z}_{12}(k) = \underset{\sim}{I}(k)k^{-1}\sinh kz_{<}\exp(-kz_{>}) \tag{31}$$

where $z_<$, $z_>$ are the lesser and greater of the depths z_1 and z_2.† An equivalent expression in terms of depth separation $\Delta z = z_> - z_<$ is

$$\underset{\sim}{Z}_{12}(k) = \frac{1}{2}\underset{\sim}{I}(k)k^{-1}\exp(-k\Delta z)[1-\exp(-2kz_<)]. \tag{32}$$

Autospectra

The displacement autospectra are a one-parameter family

$$\underset{\sim}{Z}_1(z) = \frac{1}{2}\underset{\sim}{I}(k)k^{-1}[1-\exp(-2kz)] \tag{33}$$

directly related to $\underset{\sim}{I}(k)$. The asymptotic limits

$$\underset{\sim}{Z}_1 \simeq \begin{cases} z\underset{\sim}{I}(k), & k < z^{-1} \\ \\ \frac{1}{2}k^{-1}\underset{\sim}{I}(k), & k > z^{-1}, \end{cases} \tag{34}$$

indicate an increase in negative spectral slope of unity from low to high wavenumber, with the transition occurring around $k \sim z^{-1}$. The spectral densities collapse to a single value for depths greater than k^{-1}, and are proportional to z for depths smaller than k^{-1}. This behavior

---

†The form shown is for an effectively infinite bottom depth d, i.e., when $kd \gg 1$. Otherwise, replace $\exp(-kz_>)$ by $\sinh k(d-z_>)/\sinh kd$.

is illustrated in Figure 2, for an assumed excitation spectrum of the form $I(k) \sim k^{-1.5}$. The tow spectra of Charnock (1965), well known through their reproduction in the monograph of Phillips (1966), are accounted for nicely by the prediction above (see Figure 3) if we allow ourselves the freedom to adjust the level and spectral slope (-1.5) of $\underset{\sim}{I}(k)$. Of course, this is barely more than a consistency check, but we do observe a slope increase at $kz \sim 1.0$.

Vertical coherence.

The two-depth coherence spectra predicted by Eqs. (31) and (33) are

$$\underset{\sim}{\rho}_{12}(k) \equiv |\underset{\sim}{Z}_{12}| \Big/ (\underset{\sim}{Z}_1 \underset{\sim}{Z}_2)^{1/2}$$

$$= \left[\exp(-k\Delta z) \frac{\sinh kz_<}{\sinh kz_>}\right]^{1/2} . \tag{35}$$

In the low and high wavenumber limit, the behavior is approximately

$$\underset{\sim}{\rho}_{12}(k) \simeq \begin{cases} (z_</z_>)^{1/2}, & k < z_>^{-1} \\ \exp(-k\Delta z), & k > z_<^{-1}, \end{cases} \tag{36}$$

indicating moderately high coherences at low wavenumber when the depths are not too different and a rapid decline in coherence around $k\Delta z \sim 1$. The horizontal wavelength for $\underset{\sim}{\rho} = 0.5$ is about $9(\Delta z)$. See Figure 4.

Transfer spectrum.

A quantity usually used in other applications, the transfer spectrum

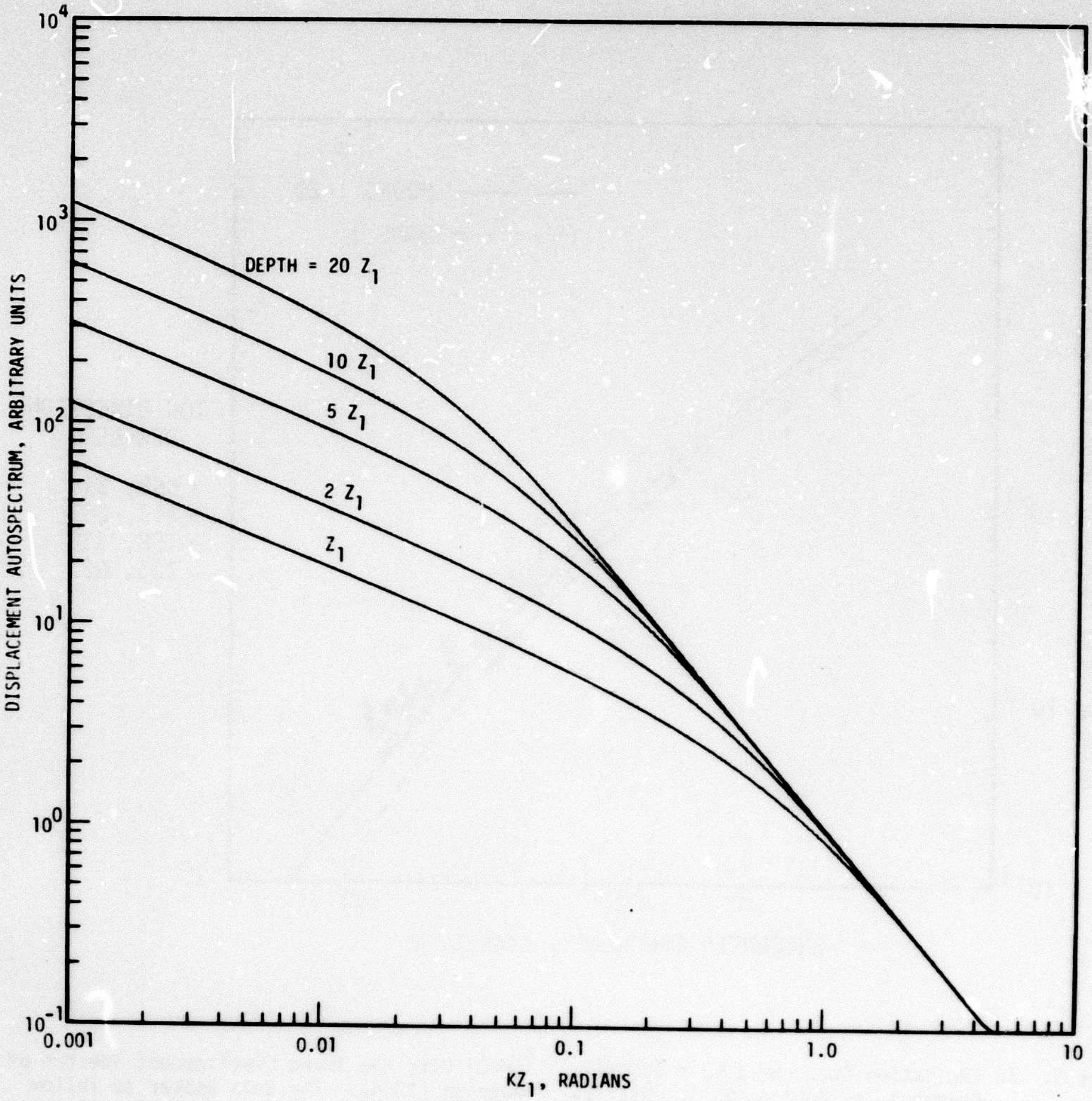

Figure 2. Spatial Autospectra of Vertical Displacement, $\underset{\sim}{Z}_1(k)$, for the Uniformly Excited Field with Spectral Dependence $\underset{\sim}{I}(k) \sim K^{-1.5}$.

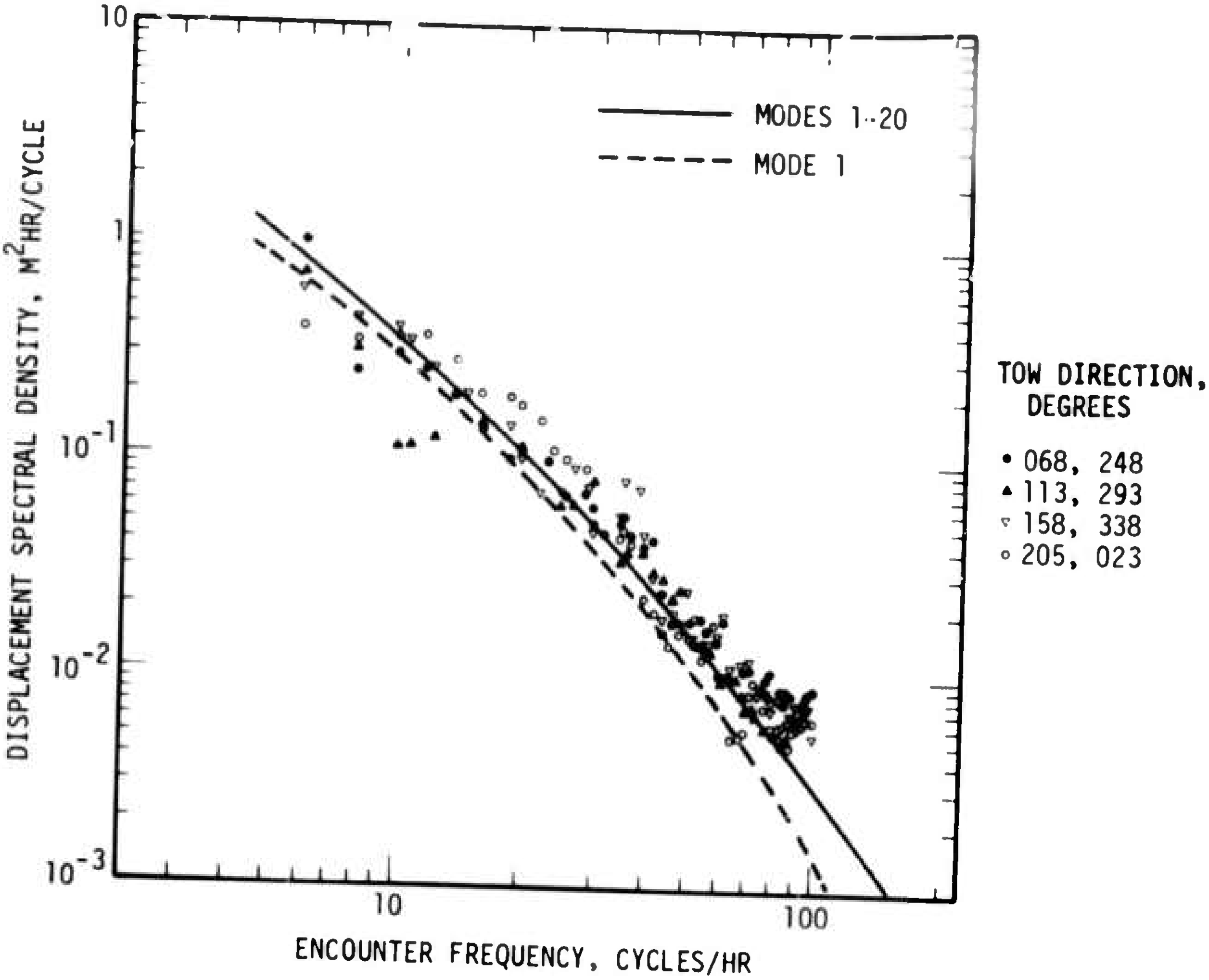

Figure 3. An Excitation Spectrum $I(k) = (0.024)k^{-1.5}$(MKS) Fits the Towed Displacement Spectra of Charnock, as Reproduced in Phillips' Monograph (1966). The data appear to follow the predicted change in slope at 45 cycles/hr, which corresponds to kz = 1 at the reported depth and tow speed of 75 m and 11.5 kt. Although the prediction is independent of N(z), the relative modal contributions are not; for the profile of Figure 1, whose maximum is close to the 75 m depth of Charnock's measurements, the first mode accounts for most of the predicted displacements.

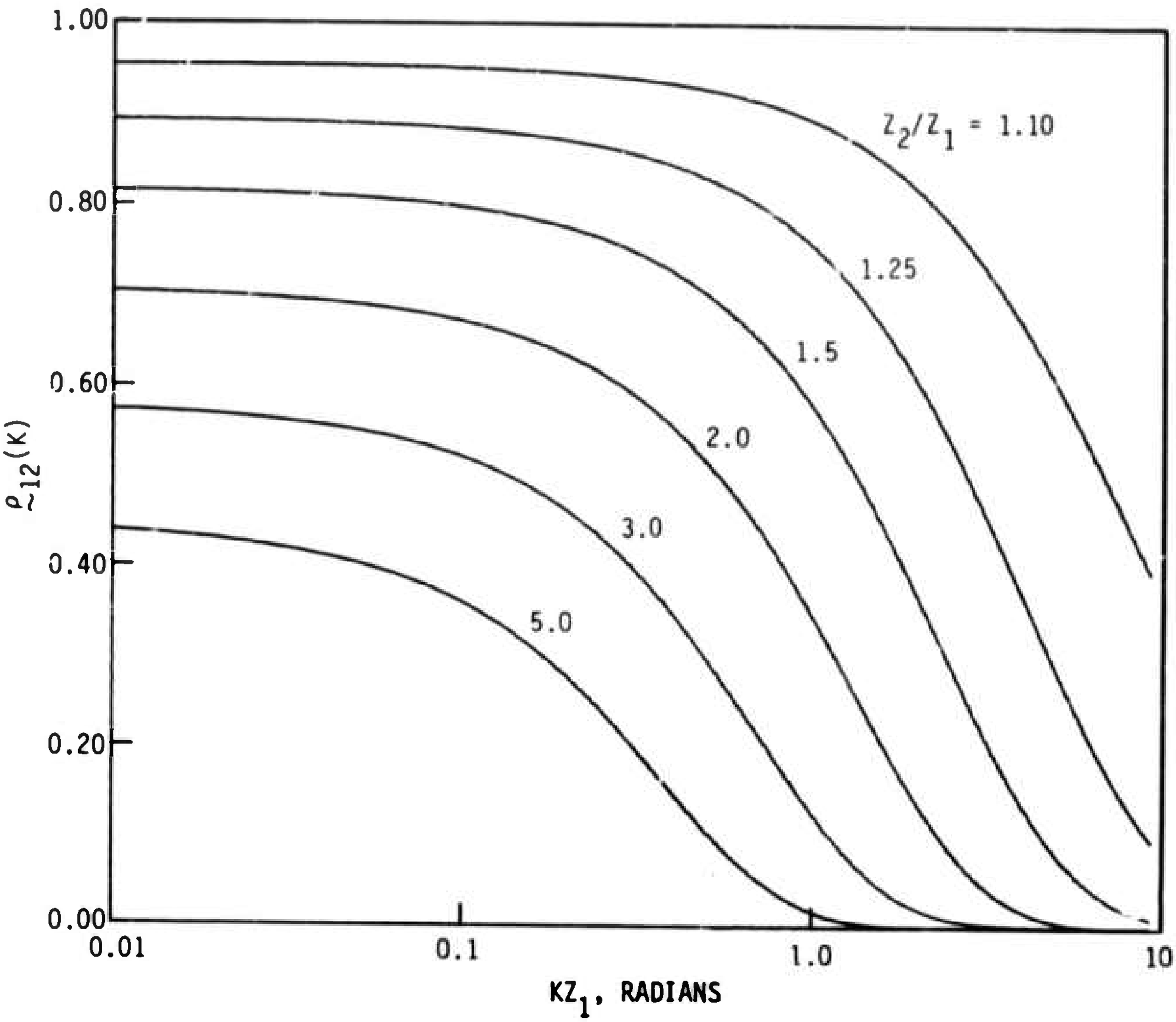

Figure 4. Spectra of Coherence, $\rho_{12}(k)$, for Spatial Series of Vertical Displacement at Two Depths in a Uniformly Excited Field. These spectra are a universal function of KZ_1 and Z_2/Z_1.

$$\underset{\sim}{I}_{12}(k) \equiv \underset{\sim}{Z}_{12}(k) \Big/ \underset{\sim}{Z}_1(k), \tag{37}$$

has a particularly simple predicted form (z_1 is the lesser depth):

$$\underset{\sim}{I}_{12}(k) = \exp(-k\Delta z). \tag{38}$$

The behavior of the multi-depth autospectra and transfer spectra predicted above will make a uniformly excited internal-wave field easy to recognize on the basis of fairly simple analysis of multi-depth towed measurements. [A towed thermistor-chain experiment has been conducted very recently (Nelson, 1975) and, for the first time, data of sufficient scope and quality for analysis in both horizontal and vertical dimensions will shortly be available.]

Treatment of towed measurements.

The spectral densities treated above, which are intended to represent distributions over *scalar* wavenumber in two dimensions, are not the same as tow spectra obtained by Fourier transform and should not be strictly compared. The expected covariance of isotherm levels at two depths z_1, z_2 and two horizontal positions x_1, x_2 along the tow,

$$C_{12}(x_1-x_2) \equiv \langle \zeta(x_1,z_1)\zeta(x_2,z_2) \rangle, \tag{39}$$

is related to the (equivalent one-dimensional) cross-spectral density $\underset{\sim}{Z}_{12}(k)$ by

$$C_{12}\,(x_1-x_2) = \frac{1}{2\pi}\iint \underset{\sim}{Z}_{12}(k)\, \exp[ik(x_1-x_2)\cos\theta]dkd\theta \tag{40}$$

where $k\cos\theta$ is the projected periodicity along the tow direction of components making an angle θ with the tow direction. Thus,

$$C_{12}(x_1-x_2) = \int_0^\infty Z_{12}(k)\, J_o(k|x_1-x_2|)dk, \tag{41}$$

and the proper inverse relation is the Hankel transform

$$Z_{12}(k) = k\int_0^\infty C_{12}(x)J_o(kx)xdx. \tag{42}$$

This indicates how true scalar wavenumber densities can be recovered from empirical linear covariances, either under the assumption of isotropy or after averaging over several tow directions.

TEMPORAL SPECTRA AND COHERENCES

The temporal spectrum $S(\omega)$ of a quantity or pair of quantities can be directly inferred from the spatial spectra $S_m(k)$ under the assumption that the modal oscillation frequencies are reasonably well represented by the dispersion relations $\omega_m(k)$ predicted by the eigenvalue equation (2). Figure 5 is a representative dispersion plot for the sample N(z) profile of Figure 1. Since the variance or covariance in the interval $d\omega$ is a sum of contributions from all modes in the proportion

$$\underset{\sim}{S}(\omega)d\omega = \sum_m \underset{\sim}{S}_m(k_m)dk_m$$

where $\omega_m(k_m) = \omega$, we have

$$\underset{\sim}{S}(\omega) = \sum_m \underset{\sim}{S}_m(k_m)\left|\frac{d\omega_m}{dk}\right|^{-1}_{k_m}. \tag{43}$$

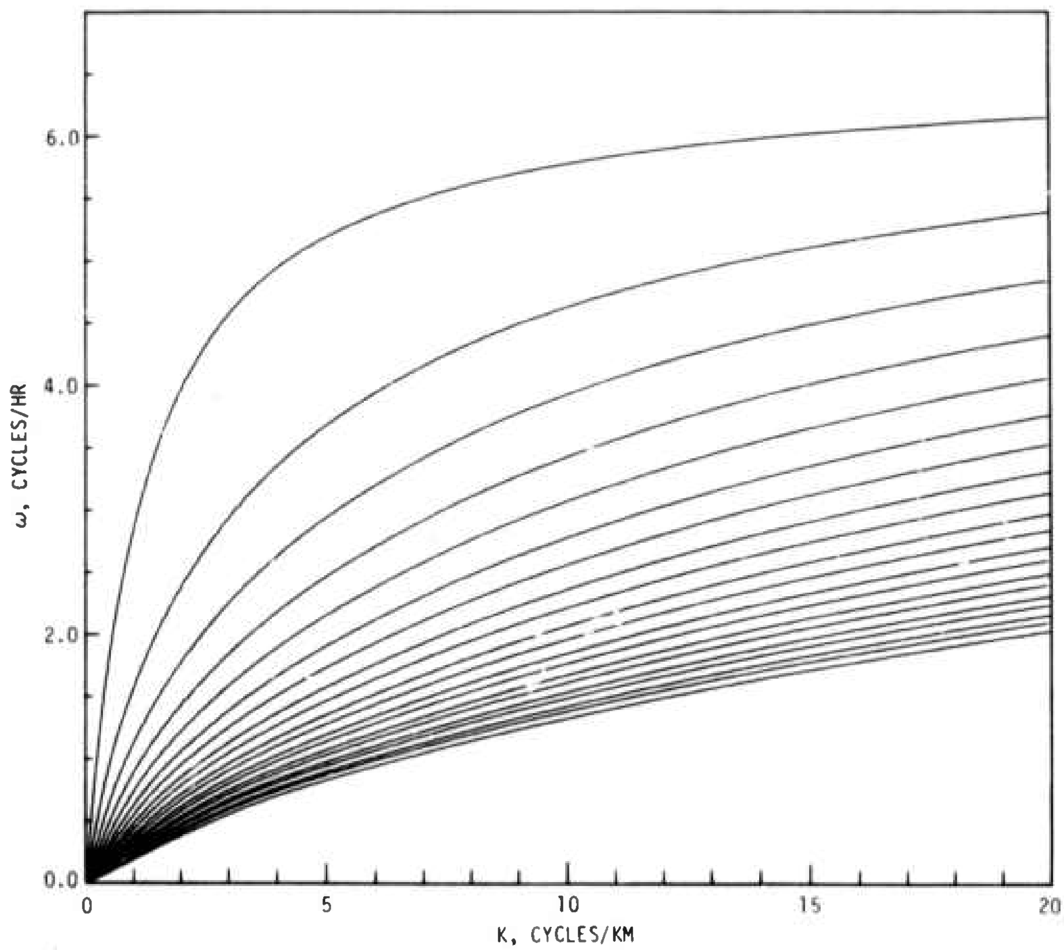

Figure 5. Modal Dispersion Frequencies, $\omega_m(k)$, for the Profile of Figure 1. These were obtained, along with the associated eigenfunctions $\phi_m(k)$, by numerical solution of the eigenvalue equation (2), and used in the sample calculations of temporal spectra.

For displacement (i.e., isopycnal) cross-spectra at two depths, the appropriate expression is

$$\underset{\sim}{Z}_{12}(\omega) = \sum_m \underset{\sim}{A}_m(k_m)c_{gm}^{-1}\,\phi_m(z_1)\phi_m(z_2) \tag{44a}$$

where c_{gm} is the modal group speed $d\omega_m/dk$, and where A_m and ϕ_m are evaluated at $k = k_m(\omega)$. In a modal sum at constant frequency, it seems preferable to use the rescaled eigenfunctions $\varphi_m(z,\omega) = (c_{gm}/c_m)^{1/2}\,\phi_m\,(z,k_m)$ which comprise an orthonormal set for the time domain, as shown in the appendix. The sum then reads

$$\underset{\sim}{Z}_{12}(\omega) = \sum_m \underset{\sim}{A}_m(k_m)c_m^{-1}\varphi_m(z_1)\varphi_m(z_2)$$

$$= \omega\sum_m k_m^{-1}\,\underset{\sim}{I}(k_m)\varphi_m(z_1)\varphi_m(z_2). \tag{44b}$$

The autospectra and coherence spectra are accordingly

$$\underset{\sim}{Z}_1(\omega) = \omega\sum_m k_m^{-1}\,\underset{\sim}{I}(k_m)\varphi_m^2(z_1), \tag{45}$$

and

$$\underset{\sim}{\rho}_{12}(\omega) = |\underset{\sim}{Z}_{12}(\omega)|\Big/[Z_1(\omega)Z_2(\omega)]^{1/2}. \tag{46}$$

Unlike the spatial spectra, these quantities depend in a subtle way on the stratification profile implicitly through $k_m(\omega)$, and no general formulae are immediately evident.

Certain qualitative features can be inferred. At low frequencies

$$c_m \rightarrow \text{const.}$$

$$\phi_m \rightarrow \text{const.}$$

so that if the excitation spectrum has the form of a power law, $\underset{\sim}{I} \sim k^{-p}$, the mode coefficients in the sums (44, 45) behave as

$$\underset{\sim}{I}_m c_m \sim \omega^{-p} c_m^{p+1};$$

thus, in the low-frequency limit, the auto- and cross-spectra have a power-law dependence with the <u>same</u> slope as the spatial autospectra at low wavenumber. Note also that for any appreciable spectral slope, say $p \sim 1.5$-2, the factor c_m^{p+1} diminishes rapidly with mode order, since $c_m \sim (m-1/2)^{-1}$, so that the sums would tend to be dominated by the first mode. This would suggest that the predicted coherences remain appreciable.

For the thermocline of Figure 1, modal eigenfunctions have been computed numerically and used to construct displacement spectra and coherence spectra corresponding to $\underset{\sim}{I}(k) \sim k^{-1.5}$. These are shown in Figure 6. The autospectra resemble some observations in that they show the expected slope and the cutoff at the local N(z); however, because of the first-mode dominance, the low-frequency values scale as z rather than as N^{-1} at shallow depth, contradicting the rule derived from the WKB approximation.

The predicted dominance of the first mode in temporal spectra means that temporal data are likely to be ineffective in determining contributions from modes higher than the first, if the distribution is truly uniformly excited. Thus, temporal data probably cannot confirm such a distribution. On the other hand, certain serious departures from uniform excitation, such as a "top hat" distribution of the kind used by Garrett and Munk, can be detected in temporal data.

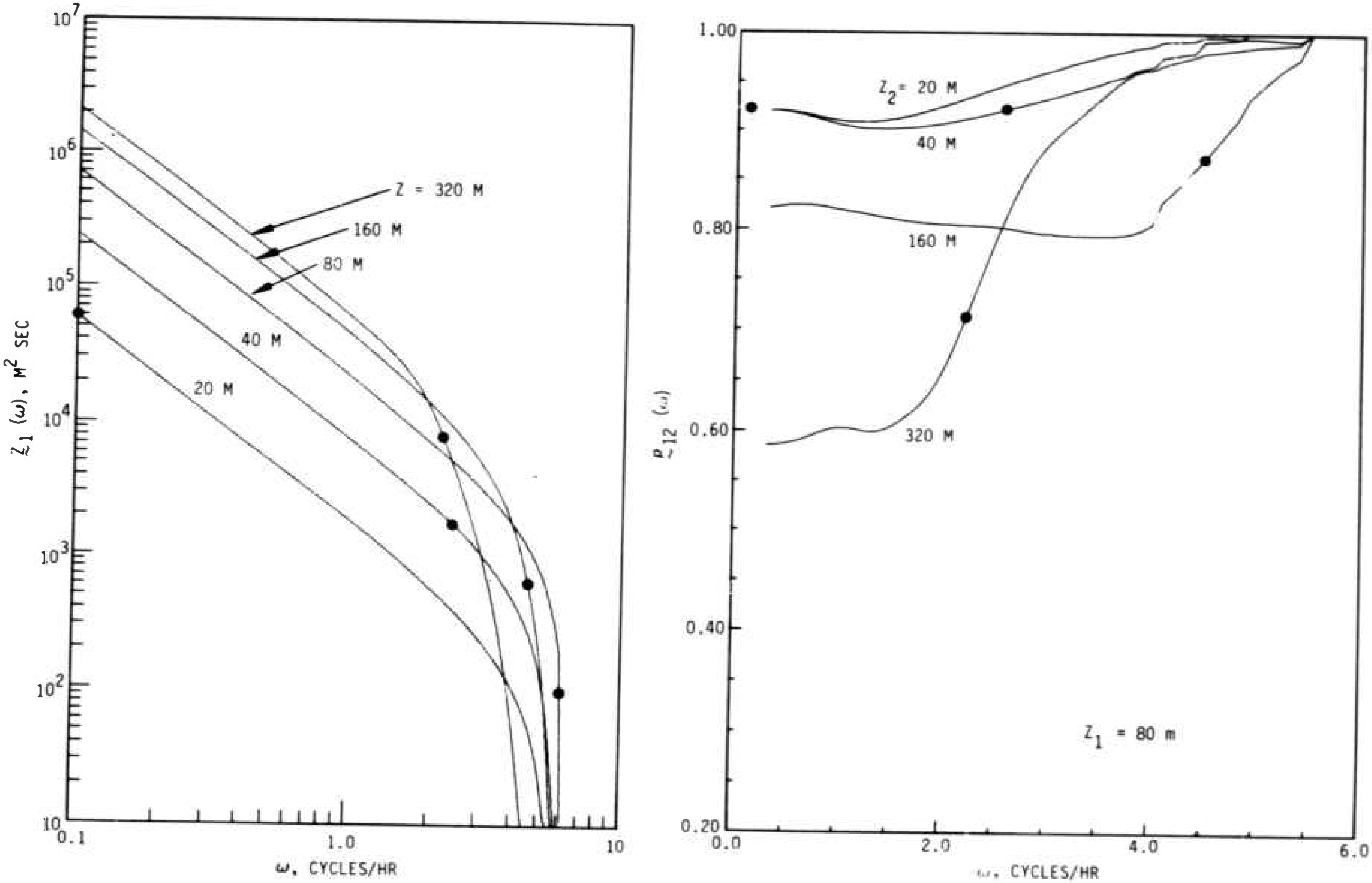

Figure 6. (A) Sample Calculation of Temporal Autospectra for Vertical Displacement at Several Depths. The circles indicate the value of N(z) at each depth for the profile of Figure 1. (B) Computed Temporal Coherence Spectra. Values rise to unity at frequencies above the local cutoff, $\omega = N(z_2)$, because the exponentially declining cross-spectrum becomes dominated by the exponential tail of the first mode. A small amount of instrument noise or fine-structure noise would drive measured coherences to zero above the cutoff frequency.

4. Comments and Comparisons

NORMAL MODES OR EQUIVALENT CONTINUUM?

Whether one discusses energy partitioning in terms of normal modes or an equivalent continuum (ω, k) is partly a matter of individual preference. Some argue that for energy at large vertical wavenumbers, the discrete normal-mode picture is, at best, a nuisance, and at worst, misleading. To the extent that the effects of shear currents are important, those normal-mode components whose phase speed is not substantially greater than the velocity shear across the thermocline are inaccurately represented. However, in the vanishing-shear approximation, the normal modes are as complete a description as any, and they have the advantage of being an orthogonal set of "degrees of freedom," independent oscillators whose excitation can be systematically compared. In retrospect they seem to be essential to a rigorous description of a "uniformly excited" field because this field has most of its energy in the lowest few modes. For example, at fixed k the first mode has as much as 75% of the total energy, and at fixed ω even more.

What is the equivalent continuum energy density for the uniformly excited field? If we define $D(\omega,k)d\omega dk$ as the energy in the entire vertical water column and in the frequency intervals $d\omega, dk$, then D is related to E_m by

$$D(\omega_m,k)\ \Delta\omega_m \simeq E_m(k) \tag{47}$$

where $\Delta\omega_m = \omega_m - \omega_{m+1}$ is the modal frequency spacing at fixed k. According to the WKB approximation, which is increasingly trustworthy at higher mode numbers, the eigenfunctions are determined by

$$k \int_{N>\omega} (N^2(Z)/\omega_m^2 - 1)^{1/2}\, dz \simeq (m - \tfrac{1}{2})\pi. \tag{48}$$

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To estimate $\Delta\omega_m$, we can differentiate by m,

$$\frac{k\Delta\omega_m}{\omega_m^2}\int N\left(1-\frac{\omega_m^2}{N^2}\right)^{-1/2} dz \simeq \pi,$$

and solving for $\Delta\omega_m^{-1}$, the "density of states" at the frequency ω, we get

$$\Delta\omega_m^{-1} \cong (\pi c_m^2 k)^{-1}\int N(1-\omega_m^2/N^2)^{-1/2}\, dz, \tag{49}$$

which, combined with $\underset{\sim}{E}_{11} = \rho c_m^2 \underset{\sim}{I}$, yields

$$D(\omega,k) \cong \rho\pi^{-1}k^{-1}\underset{\sim}{I}(k)\int N(1-\omega^2/N^2)^{-1/2}\, dz. \tag{50}$$

The continuum density obviously depends on the detailed shape of the Väisälä-Brunt profile, N(z), but if this profile is not pathological, the integral in Eq. (50) depends only weakly on ω, so that at fixed k, D is at least roughly independent of ω for frequencies distinctly less than the maximum. Further, D approaches a well-defined limit as $\omega \to 0$.

COMPARISON WITH THE DISTRIBUTION OF GARRETT AND MUNK

For simplicity, we limit the comparison to frequencies ω well below the maximum and set the inertial ı quency to zero in the Garrett-Munk expression, which allows us to assume that the dispersion curves $\omega_m(k)$ are nearly straight lines:

$$\omega_m(k) = c_m k,$$

$$c_m \simeq \pi^{-1}(M - \tfrac{1}{2})^{-1}\int N dz,$$

$$\equiv (m - \tfrac{1}{2})^{-1}c^*, \tag{51}$$

and we adjust $I(k) \sim k^{-2}$ to predict the same slope value for isotherm spatial and temporal spectra as used by Garrett and Munk. We can then compare continuum density, along with equivalent modal energy and excitation. A third distribution, one in which the energy is partitioned equally among the modes, with properties intermediate to the other two, has been added to the comparison below for purposes of discussion:

	Uniform Excitation	Energy Equipartition	Garrett-Munk '72
$D(\omega,k)$	k^{-3}	$k^{-1}\omega^{-2}$	ω^{-3}
$k^2 \underset{\sim}{E}_m(k)$	$(m-\frac{1}{2})^{-2}$	const.	$(m-\frac{1}{2})$
$k^2 \underset{\sim}{I}_m(k)$	const.	$(m-\frac{1}{2})^2$	$(m-\frac{1}{2})^3$
m	$<\infty$	$\leq j$	$\leq j$

Both the equal-energy and Garrett-Munk distributions require a mode cutoff j, which limits excitation in the (ω,k) plane to the region

$$\frac{1}{2}\omega \leq c_* k \leq (j-\frac{1}{2})\omega.$$

Note that both energy and excitation increase with mode number in the Garrett-Munk distribution, the latter quantity very rapidly.

All three distributions have been properly adjusted to exhibit the same autospectra for both stationary and towed isotherm data. Fortunately, they can be expected to predict very different vertical coherences. For example, compare the spatial cross spectra predicted for the uniform-excitation model,

$$\underset{\sim}{Z}_{12}(k) = \frac{1}{2}\underset{\sim}{I}(k)k^{-1} \sinh kz_< \exp(-kz_>),$$

with those predicted by energy equipartition ($\underset{\sim}{E}_m(k) = \underset{\sim}{E}(k)$):

$$\underset{\sim}{Z}_{12} = \rho^{-1}\underset{\sim}{E}(k) \sum_{m=1}^{j} \phi_m(z_1)\phi_m(z_2).$$

These latter become very abrupt functions of z_1-z_2 in the limit of high mode cutoff j,

$$\underset{\sim}{Z}_{12}(k) \longrightarrow \rho^{-1}\underset{\sim}{E}(k)N(z_1)^{-2}\delta(z_1-z_2),$$

suggesting an absence of measurable coherence between any but the nearest isotherms. This comparison is a particular example of how predicted coherences diminish with increasing "modal bandwidth", and indicates how the moderate bandwidth of the uniform excitation model could be distinguished from the higher bandwidth of a Garrett-Munk distribution in multi-depth data.

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APPENDIX

EIGENFUNCTIONS AT CONSTANT ω

Measurements of internal waves in the time domain are more conveniently described by a set of eigenfunctions belonging to a common frequency ω. When ω is used as an independent parameter, the eigenvalue equation reads

$$\varphi_m'' + c_m^{-2}[N^2(z) - \omega^2]\varphi_m = 0, \qquad \text{(A-1)}$$

which implies that the eigenfunctions obey the orthogonality condition

$$\int \varphi_m (N^2-\omega^2)\varphi_m \, dz = \delta_{mn}. \qquad \text{(A-2)}$$

Because of the normalization chosen at m = n, these functions are related to the previously defined eigenfunctions by some non-unit constant of proportionality,

$$\varphi_m(z,\omega) = \gamma_m(\omega)\phi_m(z,k_m(\omega)), \qquad \text{(A-3)}$$

where $k_m(\omega)$ lies on the dispersion curve

$$k_m(\omega) = \omega/c_m(\omega). \qquad \text{(A-4)}$$

(See Figure 5.) At constant frequency the eigenfunctions are complete only on the portions of the vertical domain where $N^2-\omega^2 > 0$; formally,

$$[N^2(z)-\omega^2] \sum_{m=1}^{\infty} \varphi_m(z)\varphi_m(z') = \delta(z-z'). \qquad \text{(A-5)}$$

The proportionality constant γ_m may be related to the group speeds,

$$c_{gm} \equiv d\omega_m/dk = c_m[1 + \frac{k}{c_m} dc_m/dk] \tag{A-6}$$

as follows. If we define a mode-weighted average of a function f(z) by

$$[f]_m \equiv \int \phi_m^2 f dz \Big/ \int \phi_m^2 dz, \tag{A-7}$$

then we can compare the normalization for φ_m, Eq. (A-2), with that for ϕ_m, Eq. (3),

$$\int \phi_m^2 N^2 dz = \int \varphi_m^2 (N^2 - \omega^2) dz = 1,$$

to derive

$$[N^2]_m = \gamma_m^2([N^2]_m - \omega^2)$$

or

$$\gamma_m^2 = (1 - \omega^2/[N^2]_m)^{-1} \tag{A-8}$$

Now, first order perturbation theory applied to the self-adjoint equation

$$\phi_m'' + (c_m^{-2} N^2 - k^2)\phi_m = 0$$

provides a relationship between small changes in the parameter k and eigenvalue c_m (Courant and Hilbert 1965, pp. 353-6):

$$[N^2]_m \delta c_m^{-2} - \delta k^2 = 0. \tag{A-9}$$

With a little algebra, this becomes

$$\frac{k}{c_m}\frac{dc_m}{dk} = - \omega_m^2/[N^2]_m,$$

and by (A-6),

$$c_{gm}/c_m = 1 - \omega_m^2/[N^2]_m. \tag{A-10}$$

Thus, according to (A-8),

$$\gamma_m = (c_m/c_{gm})^{1/2}. \tag{A-11}$$